Some 147/148 Level Questions (slightly above probably)

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Exercise 1 Prove or give a counterexample of the following.

(a) If $h' = g$ on [a,b], then g is continuous on [a,b].

(b) There is an increasing function with countably infinitely many points of discontinuity.

(c) If g is continuous on $[a, b]$, then $g = h'$ for some function h on $[a, b]$.

(d) Suppose that g is continuous on $[a, b]$ and $g(x) \geq 0$ on $[a, b]$. If $g(c) > 0$ for at least one $c \in [a, b]$, then $\int_a^b g > 0$.

(e) Suppose that $g(x) \geq 0$ on $[a, b]$. If $g(c) > 0$ for an infinite number of $c \in [a, b]$, then $\int_a^b g > 0$.

(f) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.

(g) There is a differentiable function on a closed interval whose derivative is not integrable.

(h) There is a function f that is integrable but there is no F such that $F' = f$.

(i) There is a function f such that $f(f(x))$ is continuous but f is not.

(j) A continuous function maps closed sets to closed sets

(k) A continuous function maps open sets to open sets.

Exercise 2 Suppose that (a_n) is a sequence of reals, all of which are in $(0, 1)$, for which the following property holds:

$$a_n < \frac{a_{n+1} + a_{n-1}}{2},$$

for all $n \in \mathbb{N}$. Show that (a_n) converges.

Exercise 3 Suppose that $\sum_{n=1}^{\infty} a_n$ is a divergent sequence of positive numbers. What can you say about $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$? What about $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2a_n}$?

Exercise 4 Suppose (a_n) is a decreasing sequence of positive reals.

(a) Show that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

(b) Generalize the previous question. Show that for any strictly increasing sequence $n_1 < n_2 < n_3 < \ldots$, for which $n_{k+1} - n_k < C(n_k - n_{k-1})$ holds for some positive constant C, $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=1}^{\infty} (n_{k+1} - n_k) a_{n_k}$ converges.

(c) Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n to \infty} na_n = 0$.

(d) Find a divergent series for which $\sum a_n$ for which the terms are positive and decreasing but $\lim_{n\to\infty} na_n = 0$.

(e) Suppose $f(x) : \mathbb{R} \to (0, \infty)$ is a decreasing function.

$$\int_1^{n+1} f\,dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^n f\,dx.$$

(f) Show that $\lim_{n\to\infty} \sum_{k=1}^{n} f(k) - \int_1^{n+1} f$ exists.

(g) Show that

$$\lim_{n\to\infty} [1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1)]$$

exists. This is called the Euler-Mascheroni constant.

Exercise 5 Show that $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ diverges.

Exercise 6 Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then the series is convergent. Also show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then any subseries $\sum_{k=1}^{\infty} a_{n_k}$ is convergent, and any rearrangement $\sum_{i=1}^{\infty} a_{\sigma(i)}$ is convergent (for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$).

Definition. A sequence x_n is said to be of *bounded variation* if $\sum_{n=1}^{\infty} |x_n - x_{n+1}|$ converges.

Exercise 7 Prove the following.

(a) Show that a sequence with bounded variation is convergent.

(b) Find a sequence which is convergent but is not of bounded variation.

(c) Show that any monotonic sequence is of bounded variation.

(d) Show that if (b_n) is a sequence of bounded variation that converges to zero, and the partial sums $\sum_{k=1}^{n} a_k$ are bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges. *This is a stronger version of Dirichlet's test.*

Exercise 8 Suppose that $\sum_{n=1}^{\infty} a_n^2$ converges. Show that

$$\limsup_{n\to\infty} \frac{a_1 + \sqrt{2}a_2 + \sqrt{3}a_3 + \ldots + \sqrt{n}a_n}{n} < \infty.$$

Exercise 9 Suppose (a_n) is a sequence of positive reals. Show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf n \to \infty \sqrt[n]{a_n} \le \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

What does this mean for the root test and the ratio test?

Kummer's Test Suppose that (a_n) is a sequence of positive reals. Let (D_n) be some sequence of positive real numbers.

(a) Suppose

$$L = \liminf_{n\to\infty} \left[D_n \frac{a_n}{a_{n+1}} - D_{n+1} \right].$$

Show that if $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose that

$$D_n \frac{a_n}{a_{n+1}} - D_{n+1} \leq 0$$

for all sufficiently large n. Suppose $\sum_{n=1}^{\infty} \frac{1}{D_n}$ diverges. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

Raabe's Test *This is the more useful test* Take $D_n = n$ to derive the following test. Compute

$$L = \lim_{n \to \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right).$$

If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ converges. If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

NOTE: This works even when the ratio test and root test fail.

(c) Let

$$a_n = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot (2n)}, n \geq 1.$$

Show that $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 10 Suppose that $f : [1, \infty)$ is a continuous, positive and increasing function with $\lim_{x \to \infty} f(x) = \infty$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

is convergent iff

$$\sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^2}$$

is convergent.

Exercise 11 Suppose (a_n) is a sequence of positive numbers, and write

$$x_n = -\frac{\ln(a_n)}{\ln(n)}.$$

Show that if $\liminf_{n \to \infty} x_n > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Show also that if $x_n \leq 1$ for all sufficiently large n, then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise 12 For positive x, find the convergence/divergence of the following.

(a) $\sum_{n=1}^{\infty} x^{\ln n}$,

(b) $\sum_{n=1}^{\infty} x^{\ln \ln n}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}}$.

Exercise 13 Show that $\sum_{n=1}^{\infty} a_n$ converges iff $\prod_{n=1}^{\infty}(1 + a_n)$ converges.

Exercise 14 Show that every continuous function mapping $[a, b]$ to itself has a fixed point.

Exercise 15 Suppose that $f : [a, b] \to [a, b]$ is continuous. Suppose that the sequence (x_n) defined recursively by $x_1 = \alpha \in [a, b]$ and $x_{n+1} = f(x_n)$ converges, then it must converge to a fixed point of f.

Exercise 16 Show that a monotone function on an interval $[a, b]$ can have only countably many discontinuities.